A Contour Integral Method for Transient Response Calculation

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A numerical method to compute the transient time-domain response using contour integration of the analytic frequency response is presented. The approach is applicable to analytic response functions that contain isolated singularities (poles) and branch points in the left half-plane. Because the method is simple to implement and requires few function evaluations, it is also convenient for computing the time-domain response of traditional polynomial filters.

0 INTRODUCTION

Although the low-frequency electroacoustic theory of Benson, Thiele and Small is primarily used to compute the steady-state pressure (SPL), velocity and acceleration of the loudspeaker as a function of frequency, one can also obtain the time-domain transient response by inverse Laplace transform. This is done analytically by Benson [1] for simple closed boxes by decomposing the response into a sum of poles such that the Laplace transform can be inverted using transform tables. In the present paper, we choose to formulate the inverse problem in terms of the dimensionless complex variable $z \doteq s/\omega_s$, where $\omega_s = 2\pi f_s$ is the resonant frequency of the loudspeaker driver (or more generally, any suitable frequency) and s is the Laplace transform variable. Then, the frequency response, $P(z)$, in the complex z-plane can be related to the step response, $p_{\rm step}(\tau)$, in the time domain via the Bromwich integral for the inverse transform (page 58, [2]):

$$p_{\rm step}(\tau) = \mathscr{L}^{-1}\left[\frac{P}{z}\right] = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} dz\, e^{z\tau}\frac{P(z)}{z} \ . \qquad (1)$$

Here, σ is chosen so that the contour lies to the right of all singularities of the integrand, as illustrated in Fig. 1. So long as the contour remains to the right of these singularities, the Cauchy integral theorem [3] guarantees that the value of the integral is independent of the path of integration. The time variable is measured in units of the dimensionless resonant time $\tau \doteq \omega_s t$.

To illustrate the inversion process for a simple case, we consider the steady-state pressure response for an undamped, closed box [1]. In this case, the response function takes the form of a 2nd-order high-pass filter

$$P(z) = \frac{z^2}{z^2 + \dfrac{z}{Q_{\rm TS}} + 1 + \alpha} \ , \qquad (2)$$

where $\alpha \doteq C_{\rm MS}/C_{\rm MB}$ is the compliance ratio and $Q_{\rm TS}$ is the driver total Q. For the special choice of $\alpha = 0$ (i.e., the infinite baffle limit $C_{\rm MB} \to \infty$) and $Q_{\rm TS} = 0.5$, we can write the inversion integral as

$$p_{\rm step}(\tau) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} dz\, e^{z\tau}\frac{z}{z^2+2z+1} \ . \qquad (3)$$

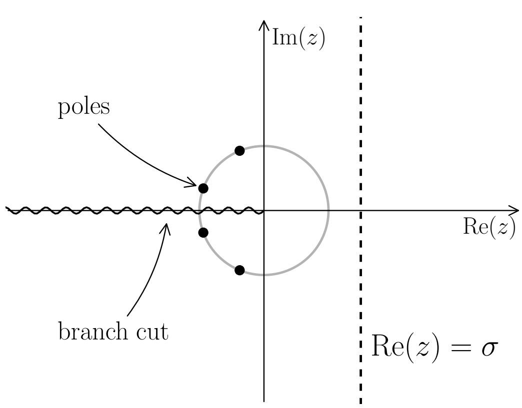

Fig. 1. Complex plane illustrating Bromwich inversion contour (dashed line) for a hypothetical case. The contour must lie to the right of all poles and branch cuts. Also shown is the unit circle, poles typical of a 4th order Butterworth filter, and a branch cut (wavy curve) at $z = 0$.

The integrand contains a pole of order 2 at $z = -1$, in which case we can close the contour to the left and use the residue theorem to give

$$p_{\text{step}}(\tau) = \frac{\partial}{\partial z}\left(ze^{z\tau}\right)\big|_{z=-1} = e^{-\tau}(1-\tau) \; . \tag{4}$$

In the more realistic case of vented and damped enclosures, the same method based on residues will work in principle, although locating the poles and computing residues may become tedious and complicated. However, in the still more complex case for which the driver circuit model contains semi-inductance [4] or creep [5, 6, 7, 8], the frequency-domain response function will contain elements described by functions with branch points – for example, $\sqrt{z}$ or $\ln(z)$. In this case, a numerical approach to evaluation of the Bromwich integral is required. The problem of numerical inversion of the Laplace transform has been an active area of research since the 1960s [9], with an important method developed by Talbot in 1979 [10]. There is no single best method; rather, the most suitable method will in general depend on the nature of the problem.

To illustrate the convergence issues associated with the inversion, consider again the example of Eq. (3). As a straightforward attempt to evaluate the Bromwich integral directly, we could put the integration contour on the imaginary axis $z = iy$ and use the property $P(-iy) = P^*(iy)$ to obtain

$$\begin{aligned} p(\tau) &= \frac{1}{\pi}\int_0^\infty dy\, \text{Re}\left[e^{iy\tau}\frac{P(iy)}{iy}\right] && (5)\\ &= \frac{1}{\pi}\int_0^\infty dy\, \frac{2y^2\cos(y\tau) + (y^3-y)\sin(y\tau)}{(1+y^2)^2} && (6)\\ &= e^{-\tau}(1-\tau) \; . && (7) \end{aligned}$$

For brevity, in the above and hereafter, we abbreviate $p_{\text{step}}(\tau)$ as $p(\tau)$. Although Eq. (7) is the same result as before, the presence of the term $\sin(y\tau)/y$ means that the integral converges very slowly. This slow convergence implies that numerical integration along a vertical contour will be an ineffective approach.

1 WEIDEMAN METHOD

Instead of the vertical contour, we will use an approach based on deforming the contour into the left half-plane. Although the basic approach is due to Talbot [10], we use the more recent optimal method due to Weideman [11], with modifications to suit the problem of loudspeaker response. It should be emphasized that application of the method requires the response function to be known as an analytic function of the complex variable z.

Weideman treats both a parabolic and hyperbolic contour, but we consider only the parabolic one:

$$z(u) = \mu\,(iu+1)^2 \; , \quad -\infty < u < \infty \; . \tag{8}$$

The integration rule is trapezoidal

$$p_{\Delta,N} = \frac{\Delta}{2\pi i}\sum_{k=-N}^{N} e^{z_k\tau}\frac{P(z_k)}{z_k}z'(u_k) \; , \tag{9}$$

where $z_k \doteq z(u_k)$, $z' = dz/du = 2\mu(i-u)$ and $u_k = k\Delta$. Although Eq. (9) provides a discrete approximation to the integral for any values of Δ and μ, Weideman shows that the optimal parameters for the parabolic contour are

$$\Delta_{\text{opt}} = \frac{3}{N} \quad \text{and} \quad \mu_{\text{opt}} = \frac{\pi}{12}\frac{N}{\tau} \; . \tag{10}$$

So long as the contour is acceptable (not passing through singularities during deformation into the parabola), the method is remarkably accurate. We remark that the algorithm can be applied for any value of $\tau > 0$, and the optimal parameters must be recomputed for every value of τ according to Eq. (10). In Ref. [11], the emphasis is on solving problems for which singularities lie on the negative real axis, in which case there are no restrictions on the smallness of μ. For τ sufficiently large, however, the magnitude of μ_{opt} will shrink so much that a pole is crossed. Since we expect poles approximately in the range $|z| \sim 1$ (for example, a Butterworth filter has all poles on the left half of the unit circle), we must ensure that $2\mu > y_{\max}$, where $y_{\max}$ is the maximum height of a pole, and 2μ is the point at which upper-half of the parabola intersects the imaginary axis. To see this, note from Eq. (8) that $z(1) = \mu(1+i)^2 = 2i\mu$.

To modify the Weideman algorithm to prevent pole crossing, it is instructive to consider in detail the behaviour of the response function for an undamped, vented box. This is written as

$$P(z) = \frac{z^4}{\left(z^2+h^2\right)\left(1+\dfrac{z}{Q_{\text{TS}}}+z^2\right)+\alpha z^2} \; . \tag{11}$$

This response function contains the new parameter $h = \omega_p/\omega_s$, where ω_p is resonant frequency of the vented enclosure. Small [12] refers to h as the *system tuning ratio*. Typical vented alignments choose $h \sim 1$. The pole locations for this response function can be computed analytically in some special cases – most notably when the denominator coincides with the 4th order Butterworth polynomial:

$$h^2 = 1, \; \alpha = \sqrt{2} \tag{12}$$

$$1/Q_{\text{TS}} = 2\cos(\pi/8) + 2\cos(3\pi/8) \doteq 1/Q_4 \; . \tag{13}$$

Note that $Q_4 \simeq 0.382683$. Then the poles occur on the unit circle at $z_k = \exp(i\theta_k)$, where

$$\theta_k = \left\{\frac{5\pi}{8}, \frac{7\pi}{8}, \frac{9\pi}{8}, \frac{11\pi}{8}\right\} \; . \tag{14}$$

In the general case, the poles can be computed using a numerical root-finding method (for example, the companion matrix method). In Fig. 2 we show the root locus for different parameter variations away from the Butterworth case. The root loci illustrate that, for a Butterworth filter, $\mu > 1$ would be more than adequate to ensure that the parabolic contour always passes over the poles. However, in the limit of large tuning ratio, $h \gg 1$, which corresponds to port tuning much higher than the driver resonant frequency, the two poles associated with the port resonance can be shown to occur at $z \sim \pm ih - \alpha/(2h)$. This behaviour is illustrated in Fig. 2c. Thus we would want to ensure that $\mu > \max\{1, h\}$.

Bearing these observations in mind, we can specify the required modifications to Weideman's algorithm as follows: choose a value of N_0 to give desired integration accuracy at short time. Using this value of N_0, define the critical parameters

$$\mu_c \doteq \max\{1, h\} \quad \text{and} \quad \tau_c \doteq \frac{\pi N_0}{12\mu_c} . \tag{15}$$

If $\tau < \tau_c$, set

$$\mu = \frac{\pi N_0}{12\tau} , \quad N = N_0 , \quad \Delta = \frac{3}{N} \tag{16}$$

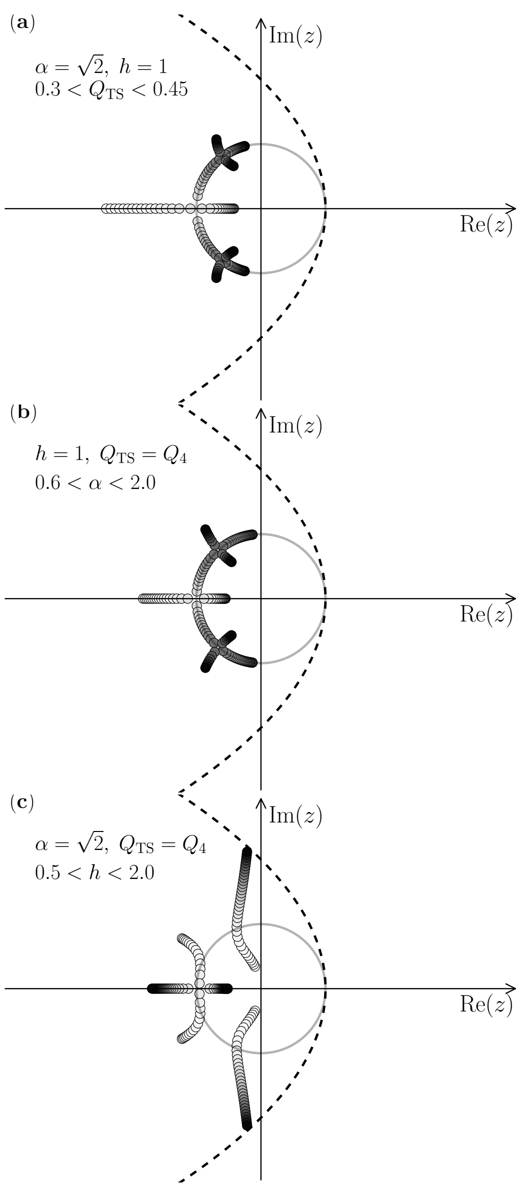

Fig. 2. Complex plane illustrating parabolic Weideman inversion contour for $\mu = 1$ and poles of the response function. Baseline parameter values are those for the Butterworth filter, as defined in Eqs. (12) and (13). Starting from the baseline values, we scan Q_{TS} in (a), α in (b) and h in (c). Plot (c) shows that large values of h can give rise to a pole crossing, which must be avoided. Darker circles indicate larger values of the scanned parameter.

Otherwise, for $\tau > \tau_c$, choose

$$\mu = \mu_c , \quad N = \left\lceil N_0 \frac{\tau}{\tau_c} \right\rceil , \quad \Delta = \frac{3}{N} , \tag{17}$$

where $\lceil\cdot\rceil$ is the ceiling function (i.e., the smallest integer greater than or equal to the argument). In other words, when $\tau < \tau_c$, we decrease μ at fixed N to stay on the optimal contour. When $\tau > \tau_c$, we must increase N to stay on the optimal contour for fixed μ_c. In practice, the method is conservative insofar as N increases more rapidly than necessary to maintain accuracy. For more complicated response functions, some method to determine the maximum height of the pole should be used, and that value should replace h in the previous formulae. Since the minimum height of the parabolic contour in the left half-plane is $y = 2\mu$, the method above is quite conservative in that it ensures the contour is at least *double* the height of the highest pole. To justify this choice, refer again to Fig. 2c. The plot shows that not only will the choice $\mu = 1$ will fail when $h > 2$, but that as the pole nears the contour, the integrand will vary rapidly, giving a large error in the trapezoidal integration scheme. Thus, the choice $\mu_c \doteq \max\{1, h\}$ will ensure the contour is well-above the highest pole.

2 APPLICATION EXAMPLES

To test the accuracy of the method for a more realistic case, we consider the previous vented box example but including the effects of suspension creep:

$$P(z) = \frac{z^4}{\left(z^2 + h^2\right)\left(\mathscr{C}(z) + \frac{z}{Q_{\mathrm{TS}}} + z^2\right) + \alpha z^2} . \tag{18}$$

Here, $\mathscr{C}$ is an analytic function,

$$\mathscr{C}(z) = \frac{1}{1 - \lambda \ln \frac{3z}{2+z}} , \tag{19}$$

that multiplies the static suspension compliance, C_{MS}. This compliance function is the so-called *three-parameter creep* (3PC) model of Ritter and Agerkvist [13], which includes both logarithmic creep, as well as frequency-dependent damping. Note that $\mathscr{C}(1) = 1$, which means that the compliance reduces to C_{MS} at $\omega = \omega_s$. We emphasize that due to the branch cut $z \in [-2, 0]$, the inverse transform cannot be computed by traditional transform tables or a straightforward integration. Figure 3 shows the time-domain step response as computed using the method of the present paper. The red and black curves, respectively, show the response with and without the creep function. We can also give an indication of the resolution required to give acceptable results. In Fig. 4 we plot the absolute inversion error as a function of the initial number of nodes, N_0. We remark that the eventually all curves overlap at sufficiently large τ for which $\mu = 1$. These results show that the present method can compute the inverse effortlessly to nearly machine precision. In fact, in generating Fig. 4, it was sufficient to use $N < 36$. This is very surprising given the apparent computational complexity of the integral in Eq. (6).

3 SUMMARY

In this paper we outline a modification to the recent Weideman method for numerical calculation of the inverse Laplace transform. The modified method is suitable for calculating the time-domain loudspeaker response and can be implemented in only a few lines of code. The complete algorithm is summarized by Eqs. (9), (15), (16) and (17), and associated formulae. Importantly, it can be applied to non-standard frequency-domain response functions containing branch cuts, as encountered in advanced transducer models with semi-inductance in the motor or viscoelastic creep in the suspension. Alternatively, this method can be used as a simple and rapid method to compute the time-domain response for simple polynomial response functions.

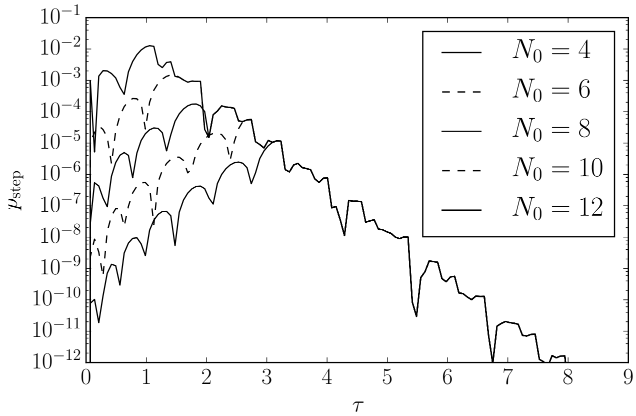

Fig. 4. Absolute error in inverse calculation for undamped vented box response function $P(z)$ (same as previous figure) with suspension creep ($\lambda = 0.5$).

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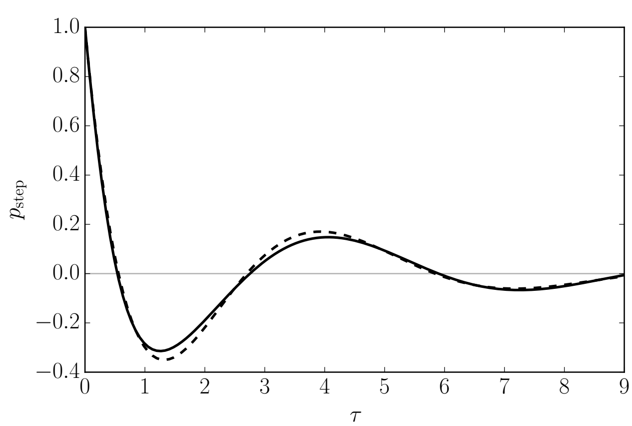

Fig. 3. Time-domain step response for $P(z)$ given in Eq. (18) with h, Q_{TS} and α corresponding to the Butterworth values. Time dependence is shown with (solid) and without (dashed) creep. The value of the creep parameter is $\lambda = 0.5$.

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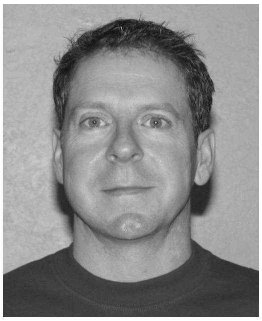

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Jeff Candy was born in Edmonton, Canada in 1966. He received his Ph.D. in Physics from the University of California, San Diego in 1994, and is currently manager of the Turbulence and Transport Group at General Atomics in San Diego, California. At General Atomics, he works on various topics in theoretical and computational plasma physics, with specific focus on plasma kinetic theory and turbulence. In the field of audio, he is interested in the application of methods in theoretical acoustics to practical situations. Dr. Candy is a member of the Audio Engineering Society and a fellow of the American Physical Society (APS).